the linearised system crosses the imaginary axis of the complex plane. An infinitesimal amplitude limit cycle oscillation (LCO) motion is expected to be created at the Hopf bifurcation as the stable fixed point loses its stability. See section 1.4 for the definitions of stable fixed point and LCO.

4.3.1 Linear Stability of the Simplified System

The simplified aeroelastic system is a system of discontinuous ODEs of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. The linear stability of a fixed point of the system can be analysed as follows.

Introducing a perturbation vector $\delta\mathbf{x}$ to the state variables representing a fixed point $\mathbf{x}$ of the aeroelastic system, the system of ODEs become

$$\mathbf{x}' + \delta\mathbf{x}' = \mathbf{f}(\mathbf{x} + \delta\mathbf{x})$$

The right-hand side can be expanded in a Taylor series to read

$$\mathbf{x}' + \delta\mathbf{x}' = \mathbf{f}(\mathbf{x}) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}} \delta\mathbf{x} + \text{h.o.t.}$$

Given that $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and ignoring the higher order terms (*h.o.t.*), the expression is simplified to

$$\delta\mathbf{x}' = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}} \delta\mathbf{x}$$

where $\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}}$ denotes the constant matrix of first order partial derivatives of $\mathbf{f}$ evaluated at $\mathbf{x}$, which is referred to as the Jacobian matrix of the system. This linear system of equations also represents the evolution of small perturbations to the system. The Jacobian matrix of the simplified aeroelastic system, denoted by $\mathbf{J}_s$, is given by

$$\mathbf{J}_s = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{16}} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_{16}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{16}}{\partial x_1} & \frac{\partial f_{16}}{\partial x_2} & \cdots & \frac{\partial f_{16}}{\partial x_{16}} \end{bmatrix} \tag{4.2}$$

The eigenvalues λ_s of the Jacobian matrix $\mathbf{J}_s$ determine the stability of the system. At the Hopf bifurcation, a pair of these eigenvalues will simultaneously cross the imaginary axis, i.e. there will be two eigenvalues of the form $\lambda_{s_{1,2}} = \pm\vartheta i$, where ϑ is a real number and $i = \sqrt{-1}$.

The expressions of the simplified aeroelastic system Jacobian matrix are given in Appendix B.

4.3.2 Linear Stability of the Full System

The full aeroelastic system can be described as a system of implicit ODEs of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{x}')$. Applying the same derivation used in the previous section one obtains

$$\begin{aligned} \delta\mathbf{x}' &= \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{(\mathbf{x},\mathbf{x}')} \delta\mathbf{x} + \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}'}\right|_{(\mathbf{x},\mathbf{x}')} \delta\mathbf{x}' \\ \delta\mathbf{x}' &= \left[\left(\mathbf{I} - \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}'}\right|_{(\mathbf{x},\mathbf{x}')}\right)^{-1} \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{(\mathbf{x},\mathbf{x}')}\right] \delta\mathbf{x} \end{aligned}$$

This is the evolution equation of the small perturbations to the system at $(\mathbf{x}, \mathbf{x}')$. The Jacobian matrix of the full aeroelastic system is given by

$$\mathbf{J}_f = \left(\mathbf{I} - \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}'}\right|_{(\mathbf{x},\mathbf{x}')}\right)^{-1} \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{(\mathbf{x},\mathbf{x}')} \tag{4.3}$$

where $\left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{(\mathbf{x},\mathbf{x}')}$ is the Jacobian matrix of the simplified system and $\left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}'}\right|_{(\mathbf{x},\mathbf{x}')}$ is the matrix of partial derivatives of $\mathbf{f}(\mathbf{x}, \mathbf{x}')$ with respect to $\mathbf{x}'$. The eigenvalues λ_{f} of the Jacobian matrix $\mathbf{J}_f$ will determine the stability of the full aeroelastic system. Finally, the Hopf bifurcation of the full system can be found by determining the value of the bifurcation parameter where a conjugate pair of λ_{f} cross the imaginary axis.

The expressions of the full aeroelastic system Jacobian matrix are given in Appendix B.