Lecture 9 – Unsteady Flow and Dynamic Stall Model

First, let us recall that we are investigating aerodynamic effects during a dynamic aeroelastic phenomenon. This involves a continuous change in the angle of attack and the subsequent flow unsteadiness. The unsteady flow physics results in the delay of flow separation as discussed earlier.

Here we will focus on the mathematical modeling of the aerodynamic forces produced under unsteady flow conditions. The method provided is a part of Leishman-Beddoes dynamic stall model.

Indicial Response Functions

An indicial response is the response signal produced by a step input into the system, i.e. for an aeroelastic system this may refer to a step change in the angle of attack and the corresponding change in lift coefficient.

The aerodynamic indicial response consists of two parts namely

1. circulatory part – which increases from zero from the moment the step change occurs and asymptotes to the final value as time increases
2. impulsive part (noncirculatory part) – which decreases from unity at the time of input step change to zero as time increases

Each part of the indicial responses will be calculated separately.

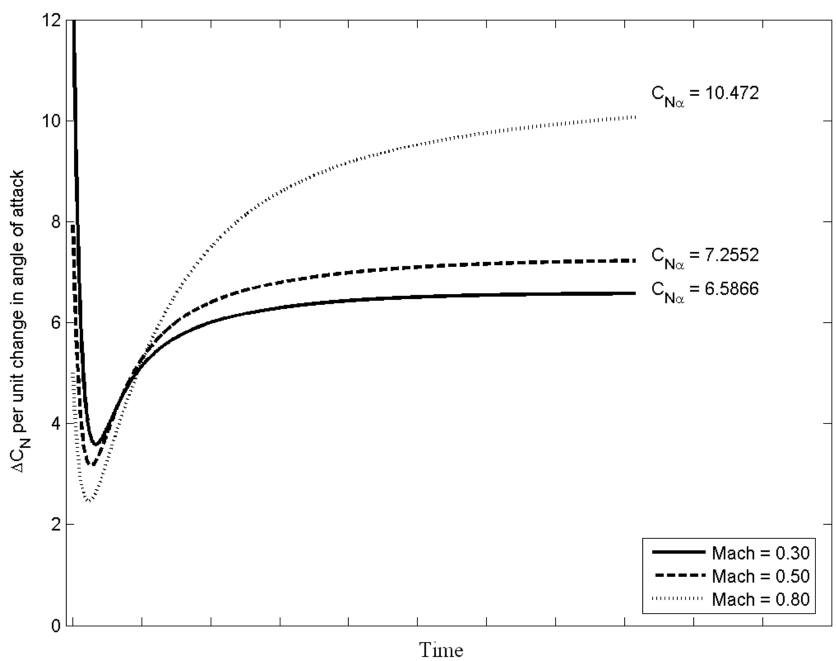

Figure 1 – Indicial C_N responses from a step change in α at different Mach numbers

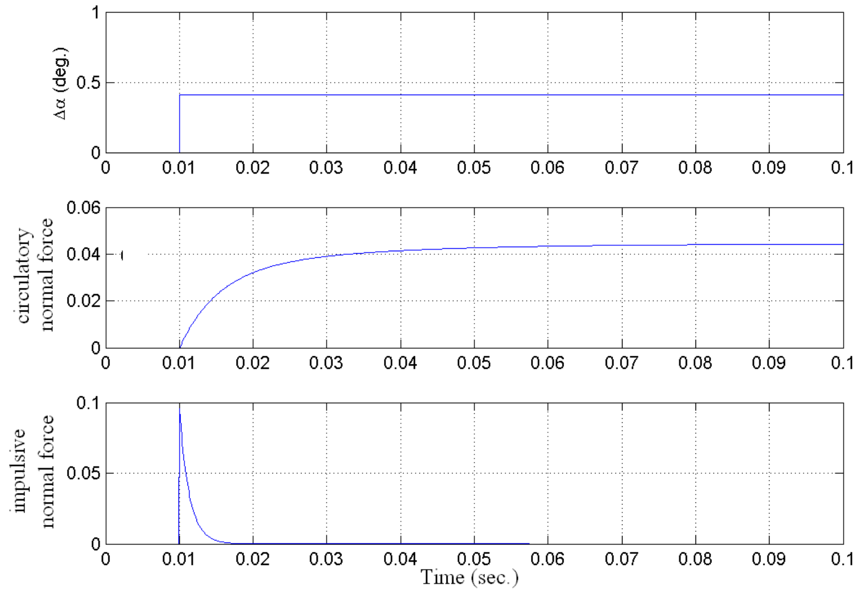

Figure 2 – Circulatory and impulsive components of the normal force responses

Mathematical Model

A change in normal force, ΔC_N, which is a result of a unit change in angle of attack, $\Delta\alpha$, is given by

$$\Delta C_N(t) = \left(\frac{4}{M}\phi_\alpha^I + C_{N\alpha}\phi_\alpha^C\right)\Delta\alpha \qquad \text{[EQN.1]}$$

where M denotes the Mach number of the flow
$C_{N\alpha}$ denotes the normal force curve slope (similar to lift curve slope)

ϕ_α^I denotes the impulsive part of the response due to a step change $\Delta\alpha$ and its formula is given by

$$\phi_\alpha^I = e^{\left(\frac{-t}{K_\alpha T_I}\right)} \qquad \text{[EQN.2]}$$

ϕ_α^C denotes the circulatory part of the response due to a step change $\Delta\alpha$ and its formula is given by

$$\phi_\alpha^C = 1 - A_1 e^{\left(-b_1\beta^2\frac{2V}{c}t\right)} - A_2 e^{\left(-b_2\beta^2\frac{2V}{c}t\right)} \qquad \text{[EQN.3]}$$

where $K_\alpha = \dfrac{0.75}{(1-M) + \pi\beta^2 M^2 (A_1 b_1 + A_2 b_2)}$

$T_I = \dfrac{c}{a}$

$\beta = \sqrt{1 - M^2}$

$A_1 = 0.30$	$b_1 = 0.14$
$A_2 = 0.70\ (= 1 - A_1)$	$b_2 = 0.53$
$A_3 = 1.5$	$b_3 = 0.25$
$A_4 = -0.5\ (= 1 - A_3)$	$b_4 = 0.1$
$A_5 = 1.0$	$b_5 = 0.5$

Table 1 – NACA0012 aerofoil indicial response properties

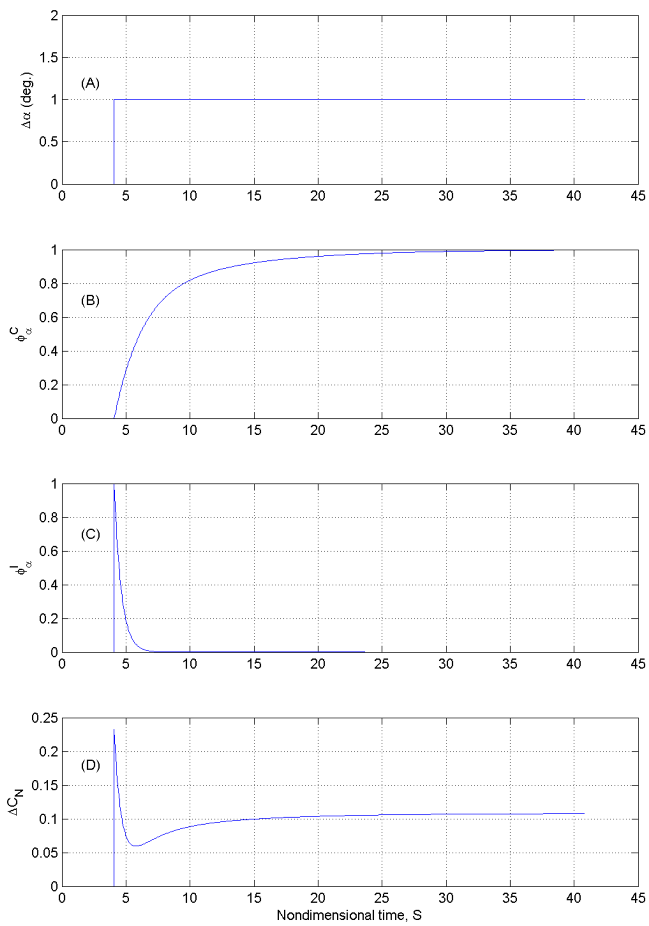

Figure 3 – Indicial normal force response functions

Superposition Indicial Responses

Step changes in angle of attack of an aerofoil against an oncoming flow are probably nonexistent in real world applications. We will almost always only experience continuous changes in the angle of attack, i.e. continuous input functions are more common than step input functions.

However, we can still apply the indicial responses to continuous inputs by using a superposition method.

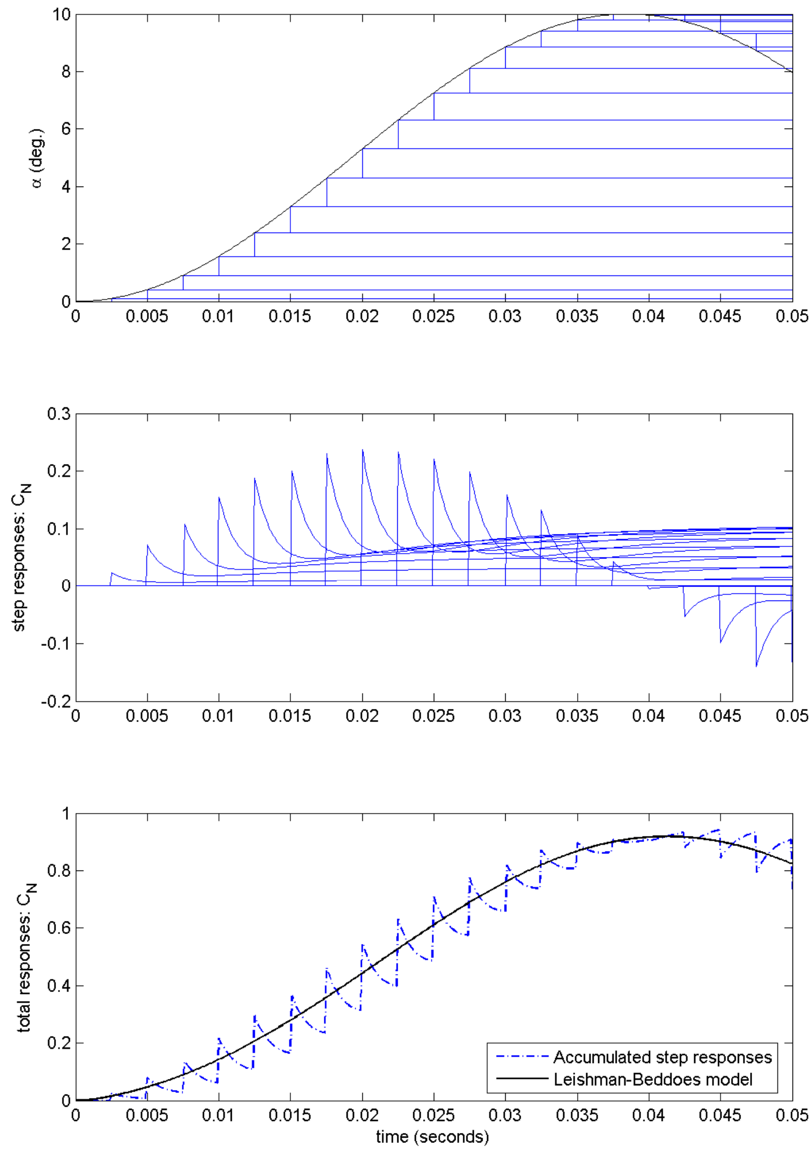

Figure 4 – Separating continuous inputs into small steps and superposition their indicial responses to obtain the total response

The total response to a continuous input can be achieved as illustrated in figure 2. The continuous input is broken into many small steps, where each step has corresponding circulatory and impulsive responses.

Finally, the indicial responses of all step changes are summed together to obtain the total response which corresponds to the continuous input. This superposition technique is called the Duhamel's method.

State-Space Method: Circulatory Normal Force

The Duhamel's method can be rewritten in the state-space form to compute the cumulative response to an arbitrary input. This formulation is widely used in control theory for analysing a multiple-input multiple-output system. The state-space formulation of an nth-order ODEs system with m inputs and p outputs is of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

where $\mathbf{x} = \{x_1 \quad x_2 \quad \cdots \quad x_n\}$ are the states of the system
$\mathbf{u} = \{u_1 \quad u_2 \quad \cdots \quad u_m\}$ are the inputs to the system, i.e. angle of attack
$\mathbf{y} = \{y_1 \quad y_2 \quad \cdots \quad y_p\}$ are the outputs of the system, i.e. normal force
A, B, C, and **D** are transfer functions

Let us consider the case of circulatory normal force due to changes in angle of attack. The system input is the angle of attack as an arbitrary function of time $\alpha(t)$, and the output is the corresponding time history of circulatory normal force $C_N^C(t)$. The circulatory normal force indicial response function in equation (3) contains two exponential terms which means that two state variables are required to simulate the same effects, i.e. a system of two ODEs is required.

Lastly, the indicial response function needs to be transformed into the recognised form of transfer functions. This is achieved by first obtaining the corresponding impulse response of the indicial function and then performing the Laplace transform on it to obtain the transfer function. For the current example, the impulse response of ϕ_α^C is given by

$$\begin{aligned} h_\alpha^C(t) &= \frac{d}{dt}\left(\phi_\alpha^C(t)\right) \\ &= A_1 b_1 \beta^2 \frac{2V}{c} e^{\left(-b_1\beta^2\frac{2V}{c}t\right)} \quad + A_2 b_2 \beta^2 \frac{2V}{c} e^{\left(-b_2\beta^2\frac{2V}{c}t\right)} \end{aligned}$$

Its Laplace transform is given by

$$\begin{aligned} \mathcal{L}[h_\alpha^C(t)] &= \int_{-\infty}^{\infty} \left[h_\alpha^C(t)\right] \; exp(-pt) \; dt \\ &= \frac{A_1 b_1 \beta^2 \frac{2V}{c}}{p + b_1\beta^2\frac{2V}{c}} + \frac{A_2 b_2 \beta^2 \frac{2V}{c}}{p + b_2\beta^2\frac{2V}{c}} \end{aligned}$$

This is the transfer function of the two state single-input single-output system. The block diagram for this transfer function is shown in figure 5. From this diagram, the state-space constant matrices associated with the circulatory part normal force and the angle of incidence can be directly determined and are given by

$$\mathbf{A}_\alpha^C = \frac{2V}{c}\begin{bmatrix} -b_1\beta^2 & 0 \\ 0 & -b_2\beta^2 \end{bmatrix}, \qquad \mathbf{B}_\alpha^C = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\mathbf{C}_\alpha^C = \frac{2V}{c}\begin{bmatrix} A_1 b_1\beta^2 & A_2 b_2\beta^2 \end{bmatrix}, \qquad \mathbf{D}_\alpha^C = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substitute these matrices into the state-space formulation, the circulatory normal force response for an arbitrary change in angle of attack is given by

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \frac{2V}{c}\begin{bmatrix} -b_1\beta^2 & 0 \\ 0 & -b_2\beta^2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\alpha(t) \qquad \text{[EQN.4]}$$

$$C_{N\alpha}^C = C_{N\alpha}^S \frac{2V}{c}\begin{bmatrix} A_1 b_1\beta^2 & A_2 b_2\beta^2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad \text{[EQN.5]}$$

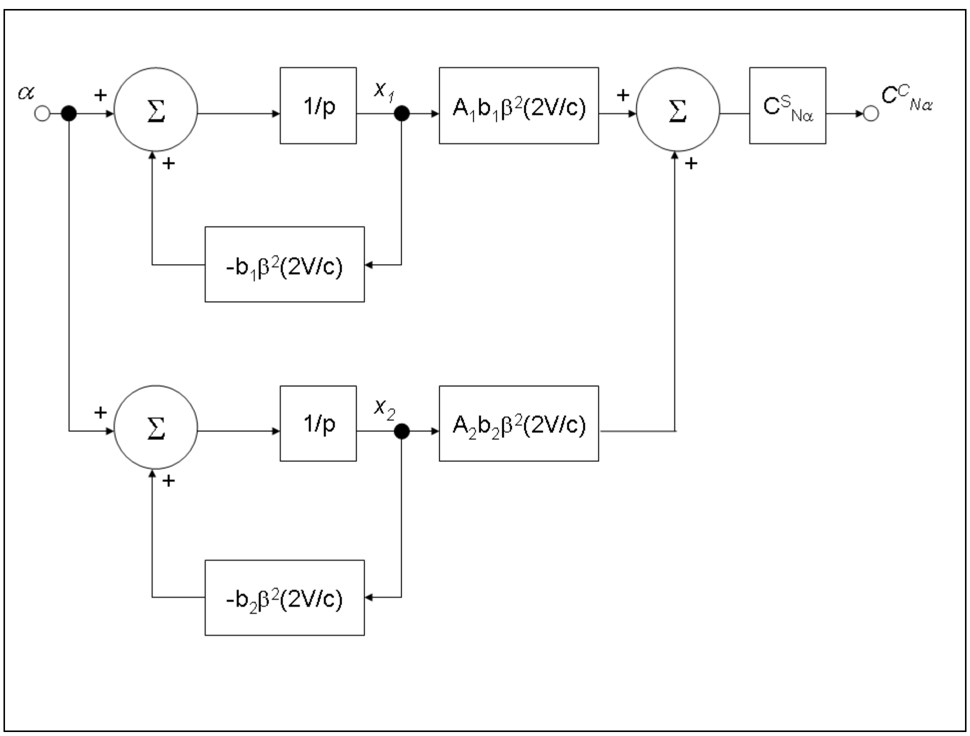

Figure 5 – Block diagram for the transfer function describing circulatory normal force response due to changes in angle of attack

State-Space Model for Unsteady Attached Flow

The complete state-space model for the unsteady flow consists of eight main components

1. Circulatory normal force due to changes in angle of attack
2. Impulsive normal force due to changes in angle of attack
3. Circulatory pitching moment due to changes in angle of attack
4. Impulsive pitching moment due to changes in angle of attack
5. Circulatory normal force due to changes in pitch rate
6. Impulsive normal force due to changes in pitch rate
7. Circulatory pitching moment due to changes in pitch rate
8. Impulsive pitching moment due to changes in pitch rate

Part 1 of the model was shown in the previous section of the lecture note.

Parts 2-8 can be derived in the same manner as shown before and are given in the supplementary materials in the Handout 9A.

The unsteady flow part can be represented by a system of eight linear ODEs, containing eight state variables x_1 to x_8.

Dynamic Stall Model

Finally, the remaining parts of the dynamic stall model are

1. Trailing edge flow separation model
2. Vortex shedding model

These components can also be written in state-space form and will contain four more state variables, i.e. $x_9 - x_{12}$. However, calculations of these state variables involve solving nonlinear ODEs and are much more complex than those in the unsteady flow part. The details of formulation of these equations are beyond the scope of this course and will not be include here. Interested readers may wish to consult this article.

State-Space Dynamic Stall Model

Finally, the normal force, chord force, and moment coefficients during a dynamic stall can be computed using these equations

$$\dot{\mathbf{x}} = f(\mathbf{x}, \alpha)$$

$$\begin{Bmatrix} C_N \\ C_C \\ C_M \end{Bmatrix} = g(\mathbf{x}, \alpha)$$