

Lecture 5: Distributed Forces

Part 1: Centre of Mass



Figure 1(a)

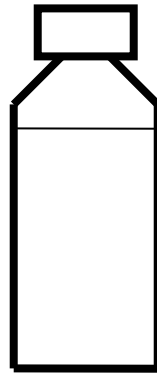


Figure 1(b)

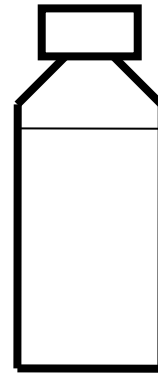


Figure 1(c)

Figure 1(a) shows a bottle of water being held aloft at its cap. The fingers are applying pressure normal to the surface of the cap while the friction between the cap and the fingers is equal and opposite to the total weight of the volume of water in the bottle.

This mechanical system is shown in _____ in figure 1(b), where the pressure applied by the fingers and the weight of the water are illustrated by _____.

Compare the system to figure 1(c), a _____ which was used before in a previous lecture. The forces are assumed to _____. The entire weight of the water in the bottle seems to act at _____. This point is its _____.

The simplification shown in figure 1(c) is a very common tool in engineering application. This is because the force at the contact between two objects almost never acts at a particular point, but over a finite area however small.

Centre of mass

The centre of mass is defined as _____.

The centre of gravity is defined as _____. For the purpose of this lecture, the centre of gravity and the centre of mass are the same and can be used interchangeably.

A simple experiment to determine the centre of mass of an arbitrarily shaped _____ (objects of constant thickness and mass, i.e. sheets of uniform materials) may be set up as shown in figure 2.

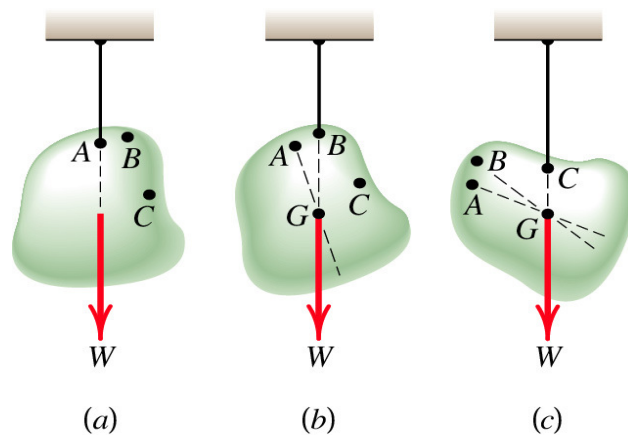


Figure 2

The centre of mass of a lamina may be found by hanging it by the small holes at different positions. The weight of the lamina must act through the centre of mass which must be vertically directly below the hole. By repeating the process several times, the point where the lines of action intersect is the position of the centre of mass.

Centre of mass of a geometric body

Here we will concentrate on the centre of mass of two-dimensional bodies of geometrical shapes, i.e. circles, rectangles, triangles, etc. The same approach may be used for one- and three-dimensional bodies.

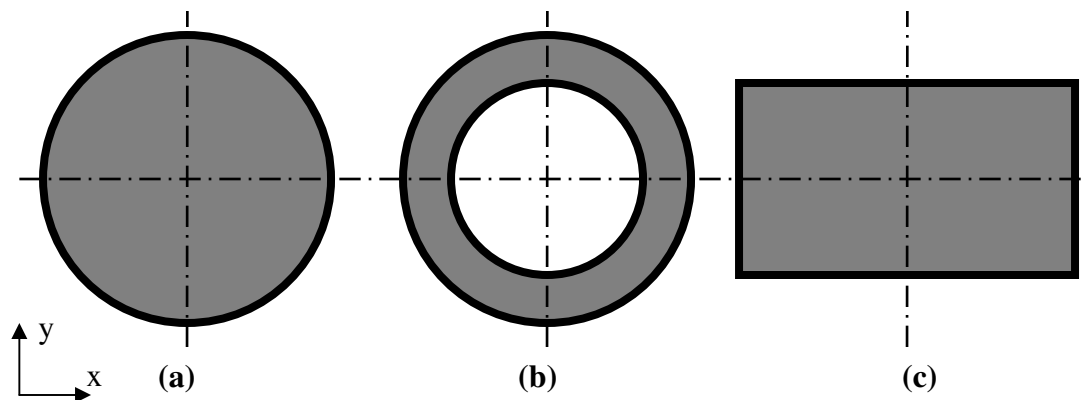


Figure 3

From inspection, the centres of mass of all the shapes in figure 3 must lie in the centres. This is because they are all symmetric about the centre lines in the x and y directions.

Position of centre of mass of a composite geometric 2-D body

For a body which consists of geometric parts such as one shown in figure 4, the position of its centre of mass may be determined by _____

_____.

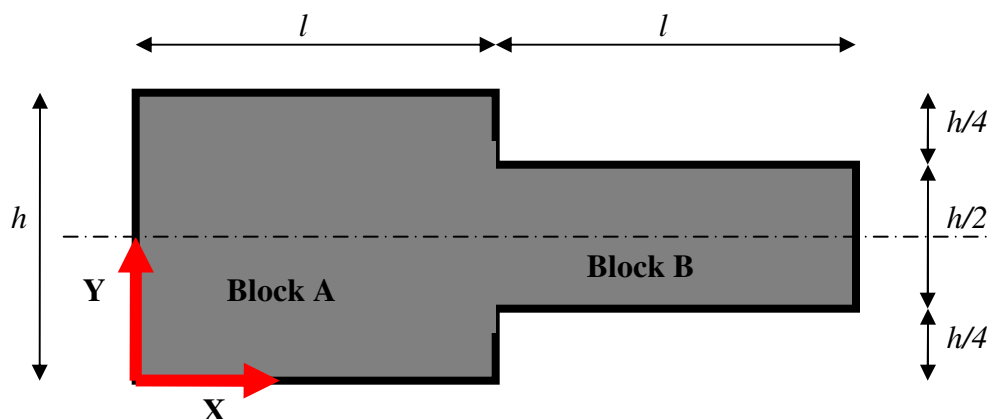


Figure 4

The coordinates of the centre of mass of block A are _____ and _____.
 Similarly, the coordinates of the centre of mass of block B are _____ and _____.

From inspection, the centre of mass of the entire body must lie on the horizontal centre line, i.e. _____, and its x-coordinate should be between those of blocks A and B.

Imagine that the block is being supported on the left and on the centre line as shown in figure 5.

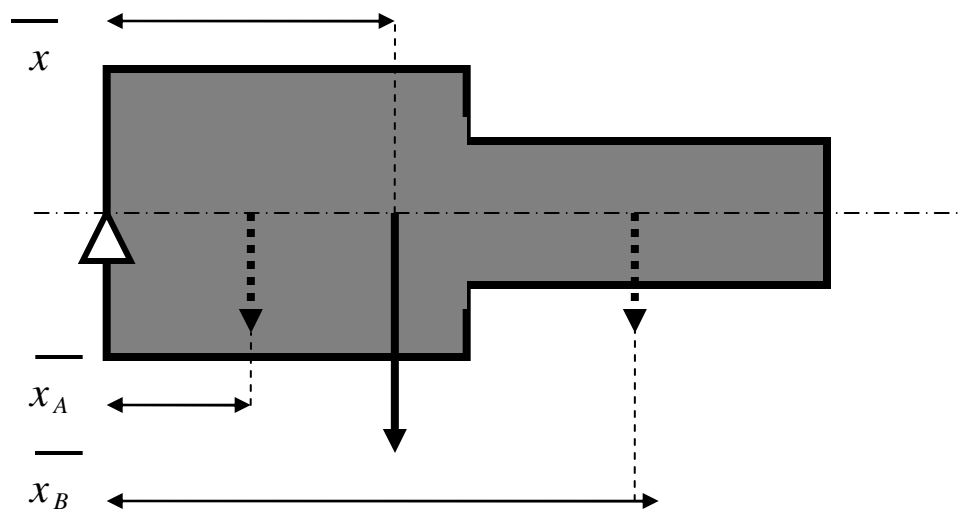


Figure 5

Consider

1. the moment generated by the weight (mass) of the _____ acting at the centre of gravity at a distance \bar{x} from the pivot
2. the moments generated by the _____

The moments generated by the two cases must be equal, hence we can derive

$$\bar{x} \sum_i^n M_i = \sum_i^n \bar{x}_i M_i$$

eqn.(1)

$$\bar{x} (\text{mass of the entire body}) = \bar{x}_A (\text{mass of block A}) + \bar{x}_B (\text{mass of block B})$$

Position of centre of mass of a generic 2-D body

For a generic two-dimensional body (lamina), the position of its centre of mass cannot be easily found using the symmetry. Consider a _____ shaped lamina shown in figure 6

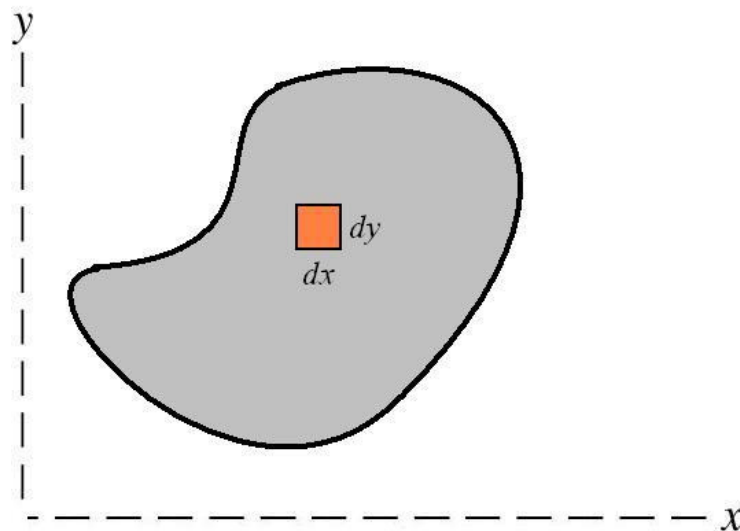


Figure 6

where A represents the area of the shaded region
 dA represents the area of the small rectangular element, in this case $dA = dy \times dx$

Recall equation (1) and using the same approach as before (lamina consists of many small elements), the x-coordinate of the centre of mass of the lamina is given by the expression

$$\bar{x}A = \sum_i^n \bar{x}_i dA_i$$

For an infinitesimally small element of area dA , the expression becomes

Similarly, the y-coordinate and the z-coordinate are given by

_____ eqn. (3)

_____ eqn. (4)

Example 1: Centre of mass of a triangular lamina

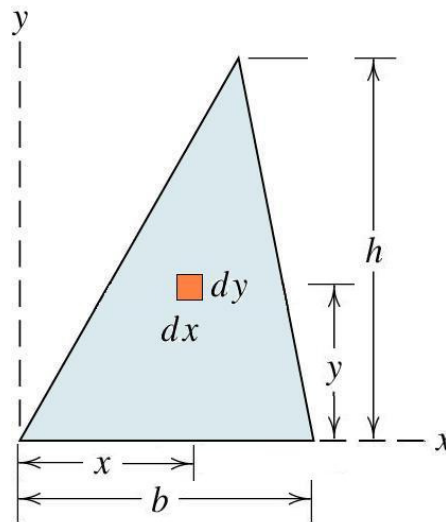


Figure 7

This example will demonstrate the determination of the coordinates of the centre of mass of the triangular shaped lamina shown in figure 7.

1. The area of the triangle is given by _____.
2. The area of the element is _____.
3. The coordinates of the element are given by ____ and ____.
4. The integration limits are given by

	Lower limit	Upper limit
x direction		: using the principles of similar triangles
y direction		

5. Using equation (3), the y-coordinate of the centre of mass is given by

$$\begin{aligned}\bar{y}\left(\frac{1}{2}bh\right) &= \int_0^h \int_0^{\frac{b}{h}(h-y)} y \, dx dy \\ \bar{y} &= \left(\frac{2}{bh}\right) \int_0^h y \left(\int_0^{\frac{b}{h}(h-y)} dx \right) dy \\ \bar{y} &= \left(\frac{2}{bh}\right) \int_0^h y \frac{b(h-y)}{h} dy \\ \bar{y} &= \left(\frac{2}{bh}\right) \left(\frac{bh^2}{6} \right) = \frac{h}{3}\end{aligned}$$

6. The centre of mass of the triangle is one-third the height of the triangle measured perpendicularly from the base. This result also holds if the calculation is made with other sides being horizontal, thus the centre of mass of a triangle lies at the intersections of the medians.

Hints and Tips

The _____ which is a result of _____, may be very difficult to compute. A simplification could be made so that the integration is less daunting. For example, the elemental area of the triangle used earlier could be changed to _____ as shown in figure 8.

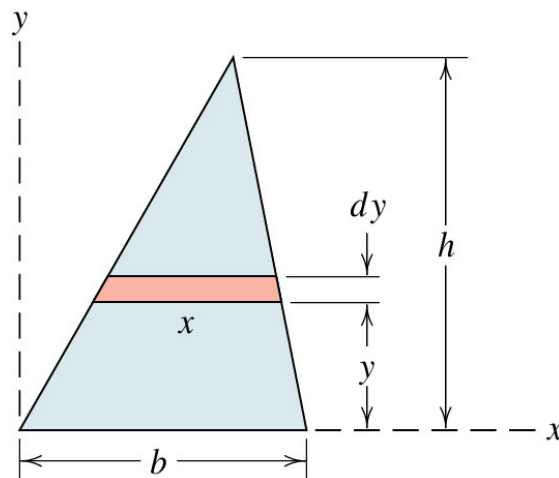


Figure 8

The area of the strip element is given by _____, where _____.

Therefore, the y-coordinate of the centre of mass can be given as

$$\bar{y}\left(\frac{bh}{2}\right) = \int_0^h y \frac{b(h-y)}{h} dy$$

which can be solved straight away.

Example 2: Centre of mass of a circular segment

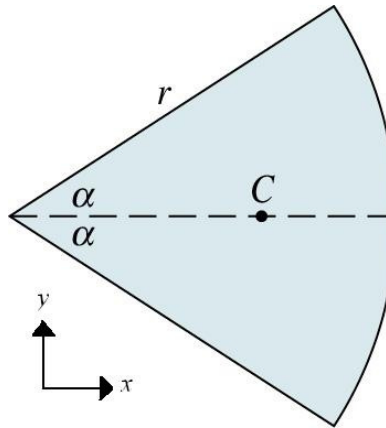


Figure 9

A segment of a circle with radius r , and angle α is shown in figure 9. The area of segment is given by _____. In order to determine the centre of mass of the segment, it is not advisable to use the Cartesian coordinates (x and y directions), i.e. $dA = dxdy$. As the integration limits become very difficult to define.

We will use the _____ in this case. Consider the strip elements shown in figure 10.

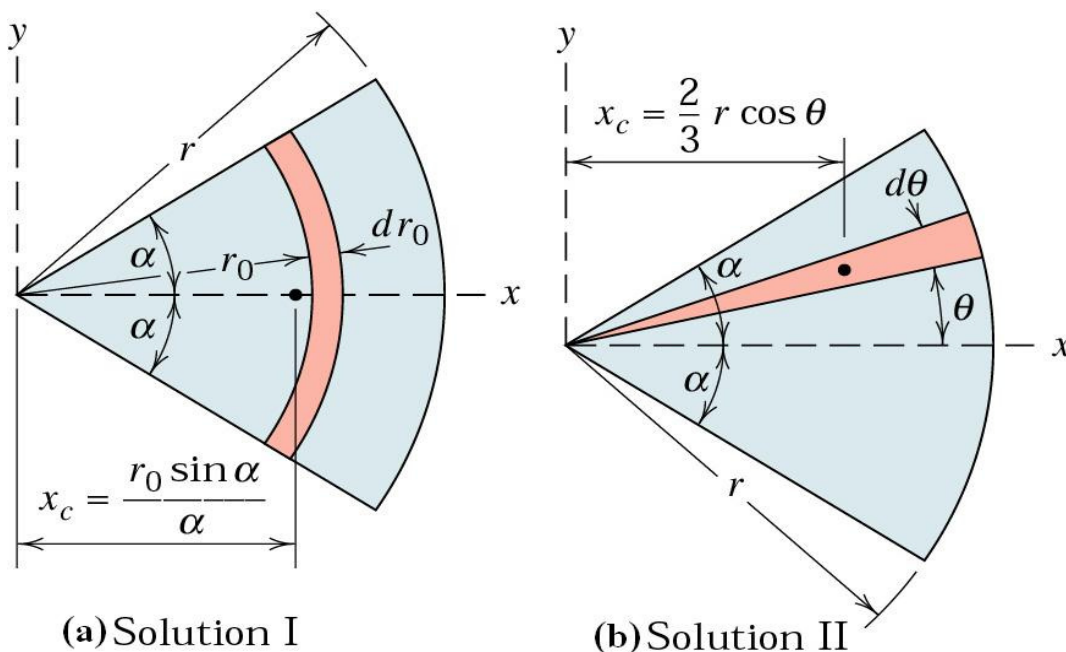


Figure 10

Case (a)

The entire segment is divided into _____ of thickness _____ whose area is, correct to first order, given by

The centre of mass of the curved strip is given by $x_c = \frac{r_0 \sin \alpha}{\alpha}$. (See Meriam page 234 for derivation)

Case (b)

The segment is divided into _____.

Each small segment is given by

The centre of mass of the small segment is given by $x_c = \frac{2}{3} r \cos \theta$.

Part 2: Second Moment of Area

The second moment of area is also sometimes called the _____.

This quantity takes the form of _____

The physical representation of the above integral can be described as follows.

When _____ are distributed continuously over _____, the _____ of these forces about some axis is proportional to the distance of the line of action of the force from the moment axis. Considering an elemental area on the surface, the elemental moment is proportional to the distance squared times the elemental area.

Example

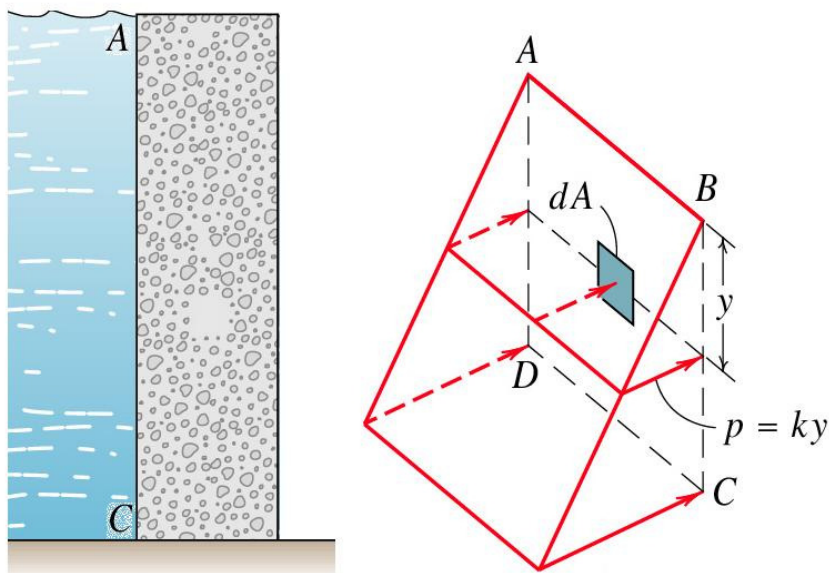


Figure 1

A reservoir wall denoted by a surface $ABCD$ is holding water on the inside. The water applies a _____ which increases linearly with the depth on the wall. The moment about the axis AB due to the pressure on the element of area dA is given by

$$dM = (py)dA = (ky^2)dA$$

Hence, the total moment over the surface $ABCD$ can be found by integrating the above elemental moment over the area. This is given by

$$M = k \int y^2 dA$$

Definitions of second moment of area

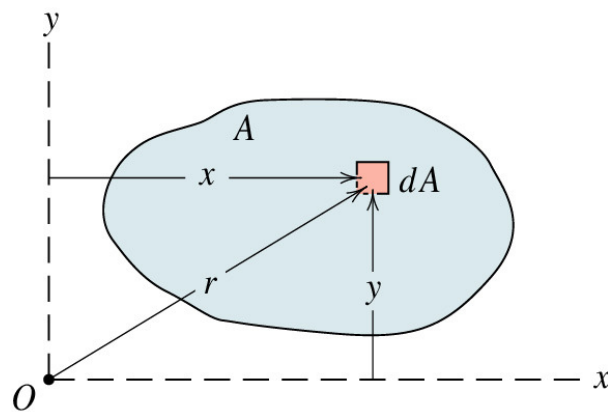


Figure 2

Figure 2 shows an area A in the x - y plane with a small element of area dA . By definition, the second moments of area of the element about the x - and y -axes are given by $dI_x = y^2 dA$, and $dI_y = x^2 dA$, respectively.

Therefore, the second moments of area of the entire area A about the x - and y -axes are defined as

Consider a rectangular lamina as shown in figure 3. We will determine the following

1. _____ (Example 1)
2. _____ (Example 2)

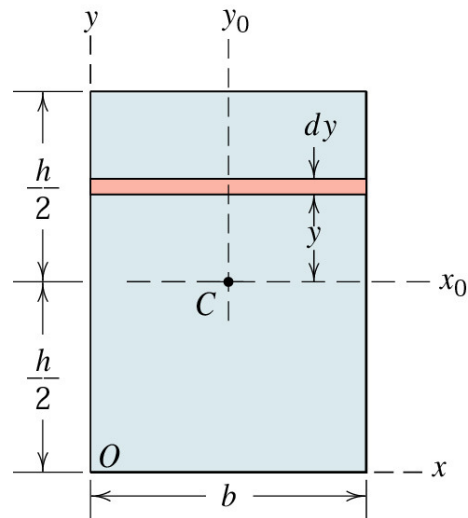


Figure 3

Example 1

We will calculate the moment of inertia about the x -axis, I_x , which lies along the bottom edge of the lamina. Let us define the elemental area dA using horizontal strip elements as shown in figure 3, so that

The integration begins _____ of the lamina and finishes at _____, so the integration limits are _____, respectively. Finally, the moment of inertia about the x -axis is given by

(1)

Similarly, the moment of inertia about the y -axis is found using vertical strip elements so the elemental area is given by

The integration is now bounded between 0 and b , therefore the moment of inertia about the y -axis is given by

(2)

Example 2

Let us now shift the x - and y -axes from the corner of the lamina to _____. From inspection, the centre of the rectangular lamina also coincides with its centre of gravity. Usually, moments of inertia taken about the x - and y -axes through the _____ are written as _____, respectively.

The moment of inertia about the horizontal axis labeled x_0 in figure 3 is given by

$$\int_A y^2 dA \quad (3)$$

Similarly, the moment of inertia about the vertical axis y_0 is given by

$$\int_A x^2 dA \quad (4)$$

Parallel Axis Theorem

Note that equations (1) and (3) are very similar. Only their integration limits are different. The difference in their physical meaning is the _____
 _____ from x_0 - y_0 (axes through the centroid).

Therefore, moments of inertia of the same body taken about axes _____
 can be related using

$$I_x = I_{x_0} + A d_x^2$$

$$I_y = I_{y_0} + A d_y^2$$

where A is the area of the lamina

d_x is the _____ between x_0 - and x -axes

d_y is the _____ between y_0 - and y -axes (See figure 4)

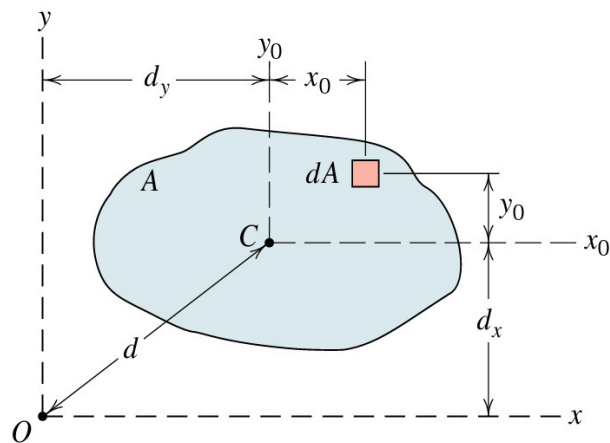


Figure 4

Radius of Gyration

The radius of gyration for the x -axis, _____, is defined by the relationship

$$\text{_____} \text{ or } \text{_____}$$

where A is the area of the lamina

I_x is the second moment of area about the x -axis (horizontal axis)

Consider an irregularly shaped lamina of area A in figure 5A. Its moment of inertia about the x -axis is given by I_x . Now, consider a long thin (negligible width) strip in figure 5B whose area is also A . The moment of inertia of the thin strip in figure 5B is also given by I_x . The distance k_x is known as the radius of gyration.

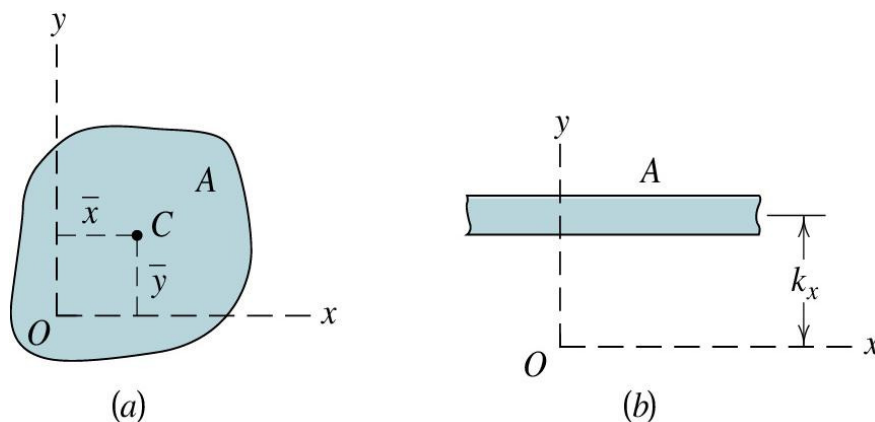


Figure 5

Similar expressions for moments of inertia about the y - and z -axes are given by

$$\begin{aligned} I_y &= k_y^2 A & \text{or} & & k_y &= \sqrt{I_y / A} \\ I_z &= k_z^2 A & & & k_z &= \sqrt{I_z / A} \end{aligned}$$

Perpendicular Axis Theorem

Suppose we need to determine the moment of inertia about the z -axis, I_z , of the rectangular lamina shown in figure 3. Let us define the z -axis to be perpendicular to the plane which contains x - and y -axes.

We can make use of known quantities found in equations (1) and (2) to determine the moment of inertia about the z -axis. The perpendicular axis theorem states that

This relationship is useful when determining the moments of inertia of a lamina about an axis which is perpendicular to its surface.

Composite Bodies

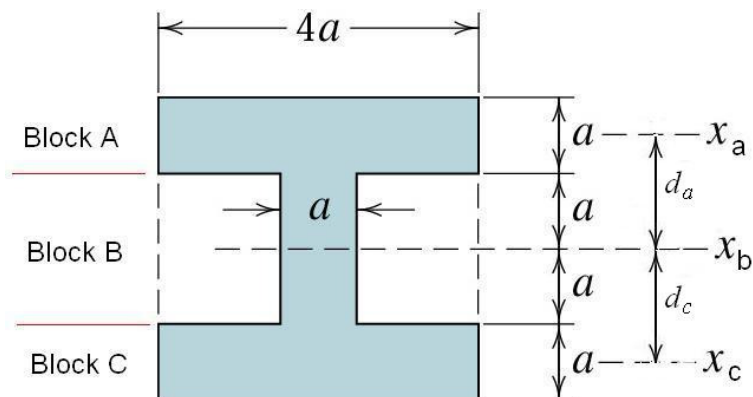


Figure 6

Figure 6 shows the cross section of an I-beam, which can be easily disassembled into three parts, block A, block B and block C, respectively. The moment of inertia of the entire shape about the central axis (x_b axis) can be found by _____ using the relationship

where d represents the perpendicular distance between the axes.

In general, the moments of inertia of a composite area about the x - and y -axes are given by

Distributed Forces – Part 3: Products of Inertia and Rotation of Axes

Products of Inertia

In certain problems involving (1) _____
and (2) _____,
and expression _____ occurs, which can be integrated to give

The quantity I_{xy} is called the _____

Note that in previous lecture moments of inertia about the x- and y-axes are called I_x and I_y , respectively. Sometimes these quantities are written I_{xx} and I_{yy} , in order to be consistent with the I_{xy} notation.

Transfer of Axes

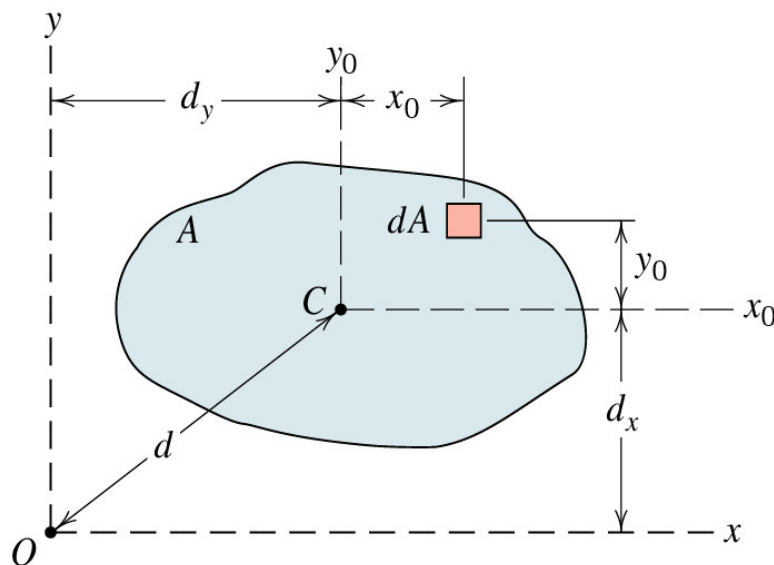


Figure 1

The axes of the product of inertia can also be transferred in a similar fashion to the _____. Since the product of inertia depends on both x- and y-

axes, the transfer of axis theorem for products of inertia for the shape shown in figure 1 is given by

where \bar{I}_{xy} is the product of inertia about the centroidal axes, d_x and d_y are the distances that the x - and y -axes have shifted, respectively.

Rotation of Axes

The product of inertia is required when we need to calculate the moment of inertia

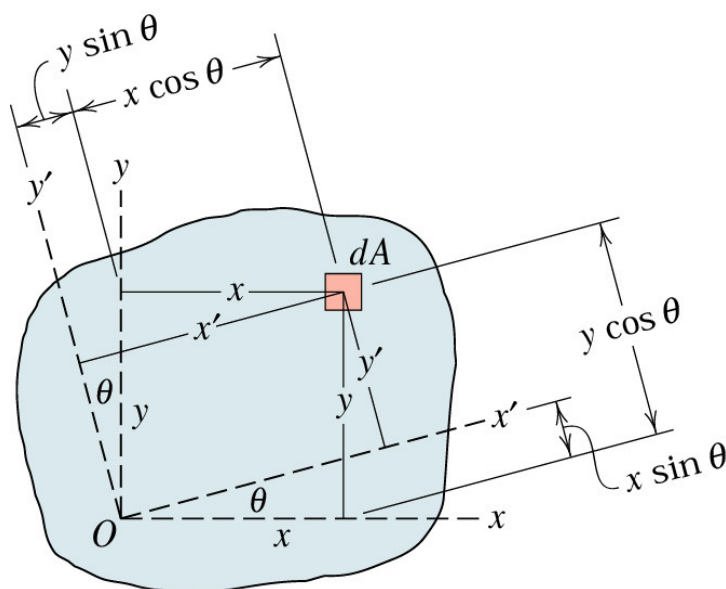


Figure 2

Figure 2 shows two sets of axes whose origins coincide on an irregularly shaped lamina. The new x' and y' axes

from the original x - y axes.

The moments of inertia about the rotated $x' - y'$ axes are given by

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad (1)$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad (2)$$

Similarly the product of inertia about the inclined axes is given by

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \quad (3)$$

Maximum and Minimum Moments of Inertia

The magnitudes of the moments of inertia about the rotated axes vary with the rotated angle θ . Hence there must exist a critical angle where the maximum or minimum value of the moment of inertia is found.

We will use a simple method from basic calculus to determine the maximum or minimum moment of inertia. By differentiating $I_{x'}$ with respect to the angle θ , and equating it to zero, we obtain

$$\frac{dI_{x'}}{d\theta} = (I_y - I_x)\sin 2\theta - 2I_{xy} \cos 2\theta = 0$$

This expression simplifies to

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \quad (4)$$

where α represents the critical angle.

Mohr's Circle of Inertia

This section will introduce another method called the _____ to determine the _____ values of moments of inertia.

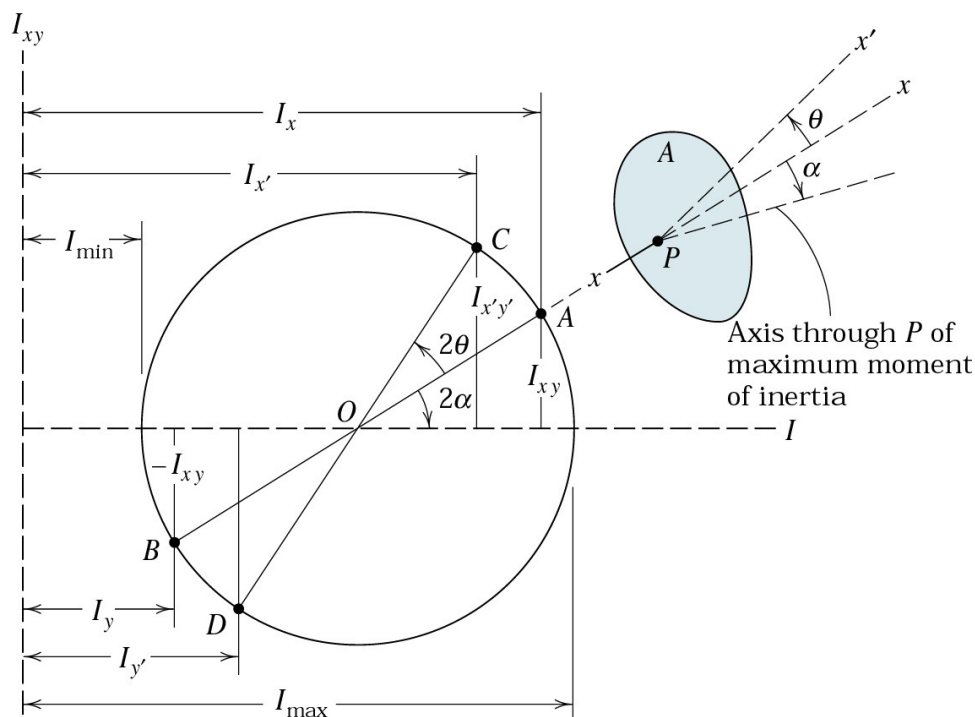


Figure 3

An example of the Mohr's circle is shown in figure 3. The circle shows moments of inertia of the shaded area A shown on the right hand side.

How do we construct the Mohr's circle?

1. The values of I_x , I_y and I_{xy} must first be defined.
2. Draw the axes using I as the horizontal axis and I_{xy} as the vertical axis.
3. Plot point A whose coordinates are (I_x, I_{xy})
4. Plot point B whose coordinates are $(I_y, -I_{xy})$
5. Draw a straight line AB . Point O is located at the intersection between AB and the horizontal axis.
6. Draw a circle whose origin is at point O and the length of its radius is the distance OA (or OB), i.e. points A and B are on the circumference of the circle.

How do we use the Mohr's circle to determine $I_{x'}$, $I_{y'}$ and $I_{x'y'}$?

Points A and B indicate the values of original I_x , I_y and I_{xy} . Suppose we want to rotate the x - y axes in the _____, see right hand side of figure 3. The new moments of inertia are now $I_{x'}$, $I_{y'}$ and $I_{x'y'}$, and the values of these quantities can be directly determined from the Mohr's circle by drawing a diameter to the circle whose inclination is at an _____ from AB .

How do we use the Mohr's circle to determine I_{\max} and I_{\min} ?

From figure 3, the points where the circle circumference intersects the horizontal axis are the locations of the maximum and minimum moments of inertia. In this case, the line OA is at an _____, which means that the axis on the shaded area which corresponds to the maximum moment of inertia is at _____.

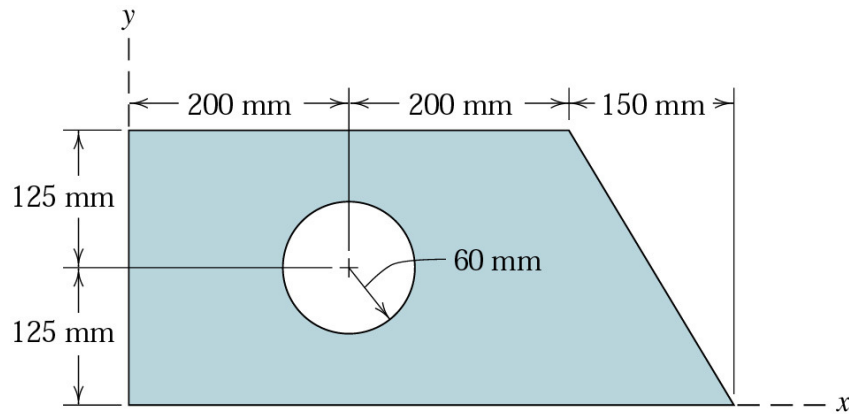
Remarks on the Maximum and Minimum Moment of Inertia

The axes where the maximum and minimum moments of inertia are found are called the _____.

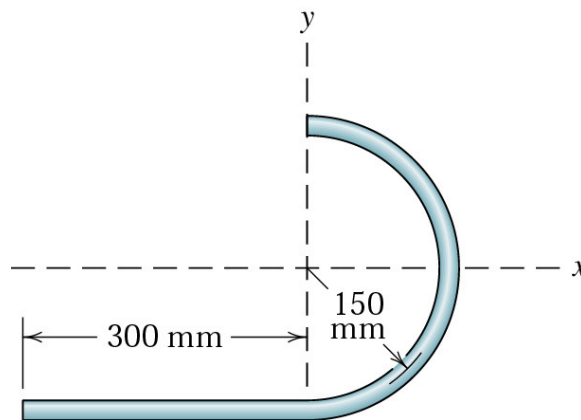
Lecture 5: Exercises

Part 1

(1) Determine the coordinates of the centre of mass of the lamina shown below (Meriam page 250)



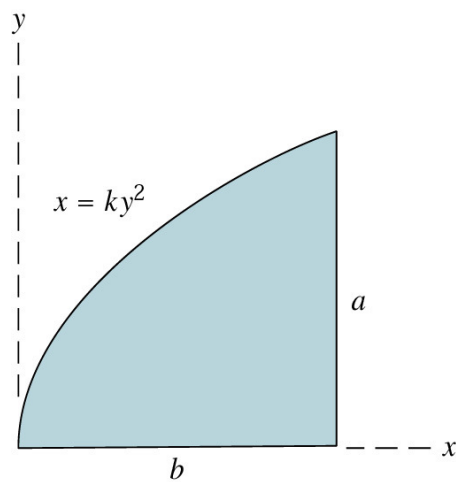
(2) Determine the centre of mass of the slender rod. (Hint: For a slender rod or a wired structure, its thickness is small compared to its length, i.e. use the length of the rod instead of the area.) (Meriam page 251)



(3) Determine the centre of mass of the segment of a circle shown in figure 10, using

- (a) ring (curved) strips
- (b) segment strips

(4) Locate the centroid of the shaded area. (Meriam page 239)



Part 2

- (1). Determine the moments of inertia about the x - and y -axes of the rectangular lamina shown in figure 3 (about point O). The elemental area must be defined as $dA = dx dy$.
- (2). Determine the moments of inertia of the I-shape lamina shown in figure 6 about the axis x_b .

Part 3

- (1) Determine the maximum moment of inertia about an axis through O and the angle α to this axis for the triangular area shown. Also construct the Mohr circle of inertia. (Meriam page 459)

