Lecture 7

Planar Kinematics of Rigid Bodies: Part 3

Relative Motion Analysis using Rotating Axes

Figure 1

The *spinning cups* ride in an amusement park consists of many rotating cups in which the riders sit. Those cups are, in turn, positioned on a large rotating plat form. Hence, from the perspective of a stationary observer, there are ______________________________ ______________________________ (1) the axis at the centre of the platform and (2) the axis at the centre of the cup.

In order to study the motion of the riders in the cup, we need to use the relative motion analysis with ______________________________ approach.

Here we will use the following sets of coordinates. The origin of the *fixed* X-Y axes is located at the centre of the rotating platform. Point A represents the centre of a cup and it also marks the origin of the rotating *x-y* axes. Point B represents a rider in the cup. Given that the cup is rotating with an angular velocity $\mathbf{\Omega}$ and an angular acceleration $\dot{\mathbf{\Omega}}$.

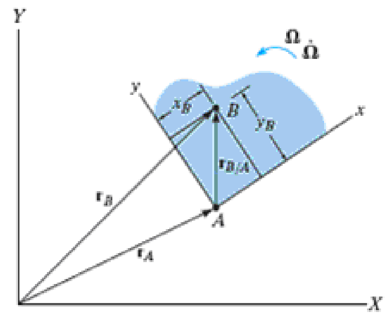

Figure 2

We will use vectors to study the motion of the rider in the cup.

Position

From the fixed X-Y axes, we can clearly define the position of the rider at point B using

________________________________ (EQN. 1)

where $\mathbf{r}_A$ is the vector connecting a fixed point to the origin (centre) of rotating axes

$\mathbf{r}_B$ is the vector connecting a fixed point to point B

$\mathbf{r}_{AB}$ is the vector connecting the origin (centre) of rotating axes to point B

Note that the fixed point used in this case is the origin of X-Y axes. However, as long as the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are measured relative to the same fixed point, equation 1 will be valid.

Velocity

The velocity of point B can be found by differentiating equation 1 with respect to time to obtain

____

The rates of change of $\mathbf{r}_A$ and $\mathbf{r}_B$ directly give the velocities of points A and B, respectively. The final term, after some manipulation, is given by

____

The subscript *xyz* denotes a ____ ____, i.e. the velocity vector $(\mathbf{v}_{AB})_{xyz}$ is the relative velocity of point *B* relative to point *A* measured from the rotating *x*-*y* axes.

Interested students may find the full derivation of the above expression on page 367 in the textbook (Hibbeler 11th Edition).

Finally, the velocity of point B is given by

____ (EQN. 2)

where	$\mathbf{v}_B$	Velocity of point *B* measured from the *XYZ* (fixed) reference
	$\mathbf{v}_A$	Velocity of point *A*, the origin of the *xyz* reference, measure from the *XYZ* reference
	$(\mathbf{v}_{AB})_{xyz}$	Relative velocity of *B* with respect to *A*, measured relative to the rotating *xyz* reference
	$\mathbf{\Omega}$	Angular velocity of the *xyz* reference, measured from the fixed *XYZ* reference
	$\mathbf{r}_{AB}$	Relative position of *B* with respect to *A*

Acceleration

The acceleration is found by differentiating equation 2 with respect to time. The derivation is not shown here but interested readers are advised to consult the textbook (page 369).

The expression for acceleration of a rigid body with rotating axis is given by

____

(EQN. 3)

where	$\mathbf{a}_B$	Acceleration of point *B* measured from the *XYZ* (fixed) reference
	$\mathbf{a}_A$	Acceleration of point *A*, the origin of the *xyz* reference, measure from the *XYZ* reference

$(\mathbf{a}_{AB})_{xyz}$, $(\mathbf{v}_{AB})_{xyz}$	Relative acceleration and velocity of *B* with respect to *A*, measured relative to the rotating *xyz* reference
$\dot{\mathbf{\Omega}}, \mathbf{\Omega}$	Angular acceleration and velocity of the *xyz* reference, measured from the fixed *XYZ* reference
$\mathbf{r}_{AB}$	Relative position of *B* with respect to *A*

Note that by comparing equation 3 to the expression for acceleration in a translating frame of reference, the additional terms that we are getting here are $2\mathbf{\Omega} \times (\mathbf{v}_{AB})_{xyz}$ and $(\mathbf{a}_{AB})_{xyz}$.

In particular, the term $2\mathbf{\Omega} \times (\mathbf{v}_{AB})_{xyz}$ is called the *Coriolis acceleration*. It represents the difference in the acceleration of B as measured from nonrotating and rotating *xyz* axes.

Planar Kinetics of Rigid Bodies: Part 1

Planar Equations of Motion of a Rigid Body

The general plane motion of a 2D rigid body consists of rotation and translation in x- and y- directions, i.e. a system with ______________________________________. This directly suggests that there are ______________________________________.

Recalling the use of ____ in previous lectures when we studied the motion of particles, it states that _________. This equation relates the externally applied planar forces to the acceleration (motion, kinematics) of the rigid body.

Finally, the rotational motion is caused by the externally applied ________________ and their relationship is given by

____

where the subscript *G* indicates that the sum of applied moment and the moment of inertia are computed at the ________________________________ of the object. (Note that detailed derivation of equation 1 can be found on page *401* of the textbook – Hibbeler 11th Edition)

Let us consider the example in the figure below. Figure 3a shows a number of externally applied forces and moments acting on the rigid body whose centre of mass is at point *G*. The *x*- and *y*- axes intersect at point *P*. These forces and moments result in a general plane motion of the rigid body as shown in figure 3b.

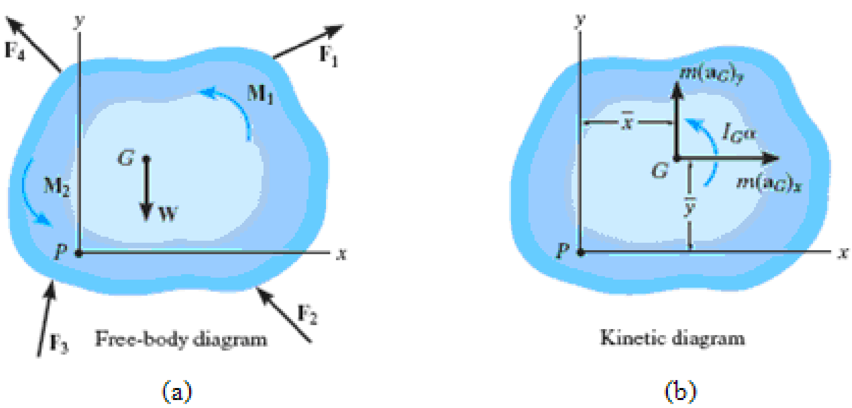

Figure 3

The equations of motion of the rigid body computed at point P are given by

____ (EQN. 4)

____ (EQN. 5)

where

$\Sigma\mathbf{F}$	The sum of force vectors
m	Mass of the rigid body
$(\mathbf{a})_G$	Acceleration of point G, $(\mathbf{a})_G = a_x\mathbf{i} + a_y\mathbf{j}$
ΣM_P	The sum of moment about point P (positive anti-clockwise)
$\bar{x}, \bar{y}$	Distances in the x- and y- directions between point G and point P, respectively
I_G	Moment of inertia about an axis normal to the x-y plane and passes through point G
α	Angular acceleration of the rigid body about point G (positive anti-clockwise)

In general, we can separate the components of equations 4 and 5, so that they can be rewritten as

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The term r represents the distance between point G and point P. The terms in the brackets represent the moment of inertia of the rigid body about the axis through point P computed using the __________________________________.

Equations of Motion: Rectilinear Translation

We can simplify equations 4 and 5 slightly if we are only considering a translation motion. For a rectilinear motion, we can use the equations

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Note that the sum of moment about the centre of mass is zero. However, the sum of moment about any other point may yield a non-zero result.

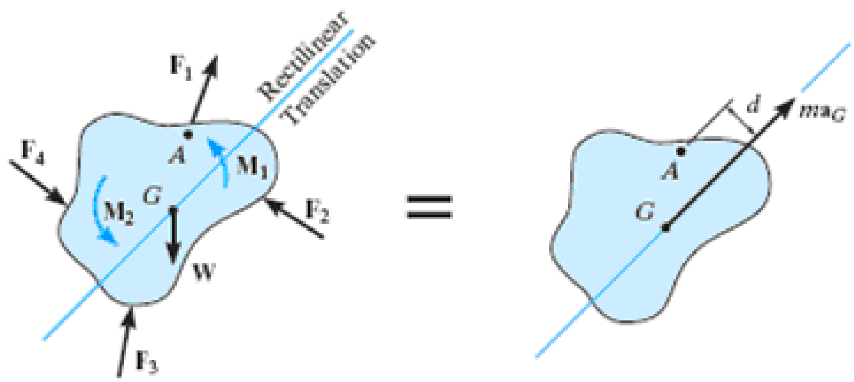

Figure 4

Equations of Motion: Curvilinear Translation

For a rigid body undergoing a curvilinear translation, it may be more convenient to work with a ____ instead of the x-y coordinates.

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A similar derivation as the rectilinear translation can be applied here. The sum of moment about the centre of mass is zero, but not at other points, i.e. there is a resultant moment measured about point B as shown in figure 5 (right hand side).

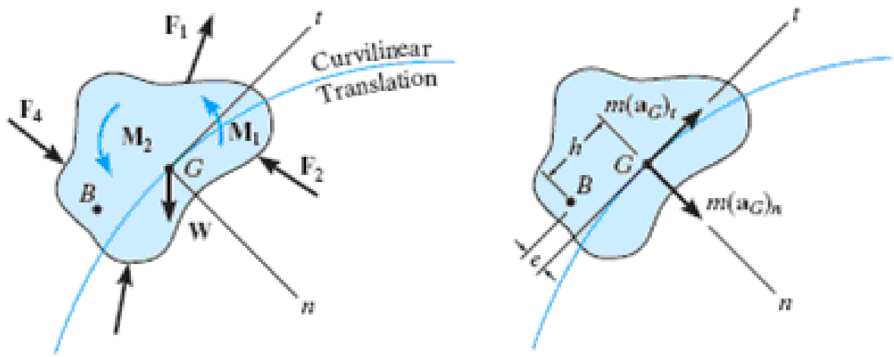

Figure 5

Equations of Motion: Rotation about a Fixed Axis

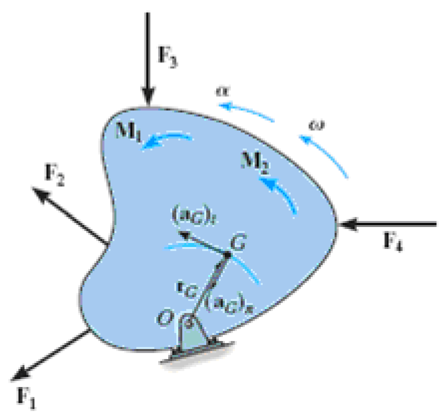

Figure 6

For a pure rotation motion about a fixed axis, it is usually most convenient to perform the calculation at the centre of rotation. In the example shown in figure below, the rigid body is rotating about the hinge at point O. The equations of motion of the rigid body in this system are given by

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The terms in the bracket in the moment equation actually represent the moment of inertia about the axis which passes through point O, i.e. ________________________.

Rotation with Friction

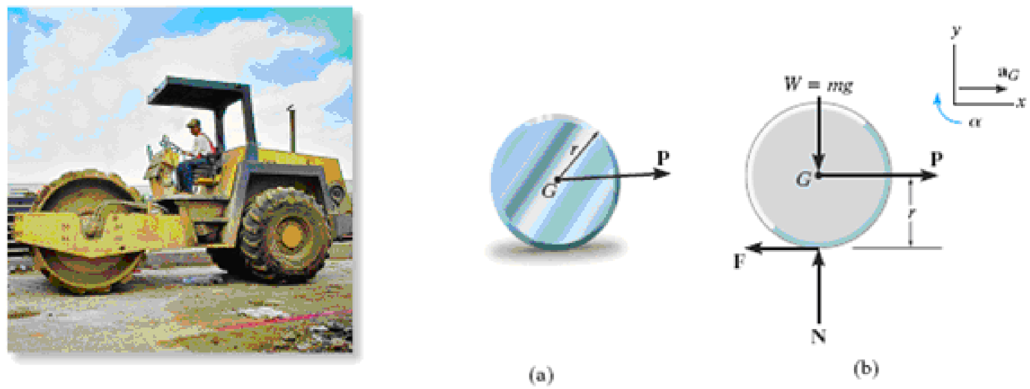

Figure 7

Let us consider a system of forces acting on one of the rear wheels of a truck. It rolls without slipping and the friction between the wheel and the ground results in the forward motion of the vehicle.

The free body diagram in figure 7b (not related to the force system of the wheel) shows a system of forces acting on the cylinder which is free to roll on the ___________________ horizontal surface. The equations of motion of this system are given by

$$\left.\begin{aligned} \Sigma F_x &= m(a_x)_G \\ \Sigma F_y &= m(a_y)_G \\ \Sigma M_P &= I_G\alpha \end{aligned}\right\} \qquad \begin{aligned} P - F &= ma_G \\ N - mg &= 0 \\ Fr &= I_G\alpha \end{aligned}$$

If we are given the applied force *P* and the physical attributes of the cylinder (*m* and *r*), there are still four unknowns with only three equations. The final equation can be formulated from the _______________________________________.

Case1 – No slipping

If the frictional force F is so high that it does not allow slipping to occur, then we may use the kinematics relationship

____

It is a good practice to double check if slipping is really not present by using the relationship $F \le \mu_s N$ where μ_s is the static friction coefficient between the cylinder and the surface it is rolling on.

Case2 – Slipping

We will have kinetic friction instead of static friction if slipping occurs. The fourth equation in this case will be given by

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