Lecture 10: Virtual Work

Work of a Force

Work done by a force in straight lines

Work of a force is a _________________quantity and it is defined by the product of the displacement that a body has traveled in the direction of the force it is subjected to. The mathematical expression of work done U by force **F** is given by

____

where **d** is the distance traveled in the direction of force **F**. Note that **F** and **d** are vector quantities and their directions must be clearly defined. Also recall that a dot product of vectors is a scalar quantity.

Non-parallel force systems

A rigid body is sometimes constrained not to travel in the direction of the force as shown in figures 1a and 1b. The block is moved a distance **Δs** due to the externally applied force **F**, which inclines at an angle α to the horizontal.

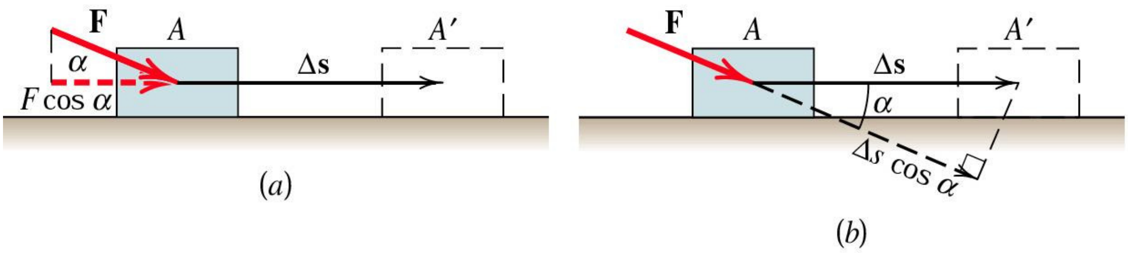

Figure 1

The work done from both systems are the same________________________________.
It shows that work done can also be defined as the product of the displacement traveled and the component of the force in the same direction.

Work done along a continuous path

With the second definition of work done by a force, let us now consider the force system in figure 2.

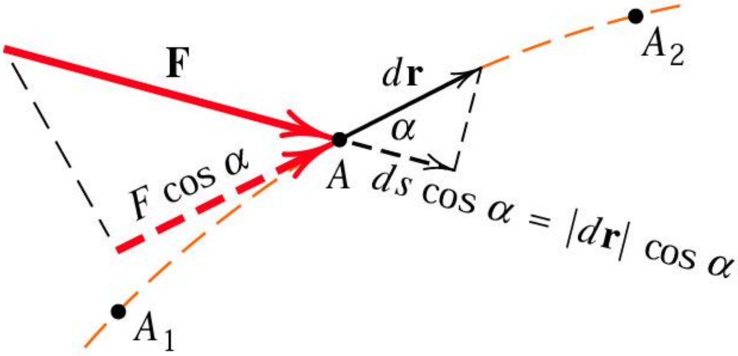

Figure 2

The force **F** is acting on a body at point *A* and this causes it to move along the path from A_1 to A_2 as shown. Since the path is not a straight line, we will look at the elemental level of the force and displacement components. From the figure, we obtain

dU = ______________________________

where F denotes the ________________________, i.e. $F = |\mathbf{F}|$

$d\mathbf{r}$ denotes the ____________________________________

The ________________________________ by F during a finite movement of point A from A1 to A2 can be found by integrating the expression above to obtain

U = ______________________________

Work of a Couple (Moment)

The resultant motion of a body due a couple may consist ____________________ ________________________________. However, there is always a pair of equal and opposite (positive and negative) work done during the translation of the body which cancel themselves out. See illustration in figure 3b.

Hence, considering an infinitesimal element, the work done by a couple is given by

____

The above expression can be integrated to obtain total work done during a finite rotation

____

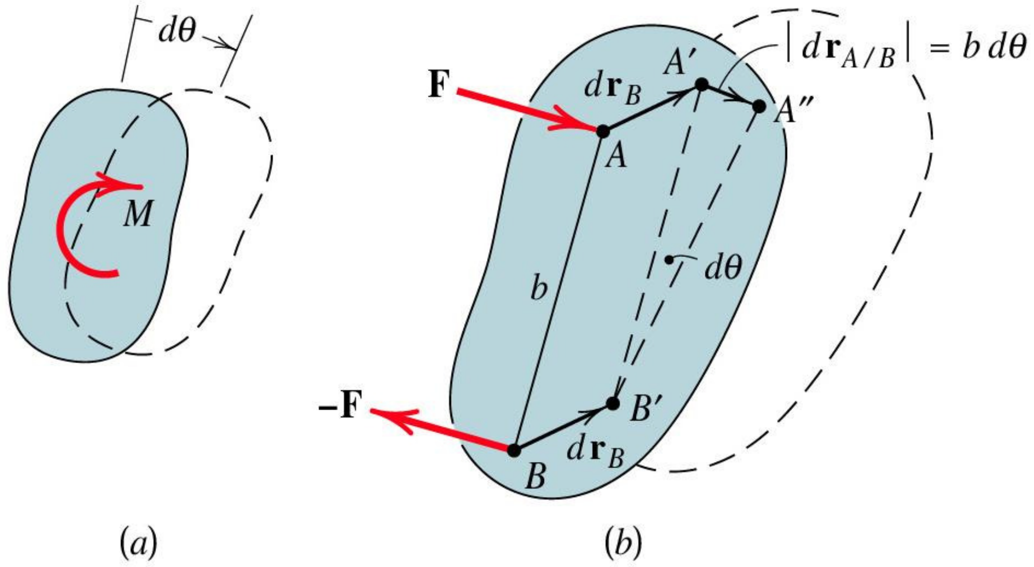

Figure 3

Principle of Virtual Work

Let us first define __________ types of forces which will be used in the analysis of ___________________________.

1. ___________________________ are ____________________________ capable of doing virtual work during possible virtual displacements. See figure 4a.
2. _____________________________ act on the structures __________________ _____________________ where no possible displacements are allowed in the direction of the force. See figure 4b.
3. ______________________________ are those compression or tension forces that the members carry. These forces always appear in equal and opposite pairs; hence their net work done is always zero. See figure 4c.

Note that only active forces can possibly produce work during any possible movement of the system, hence we will only concentrate on this type of force.

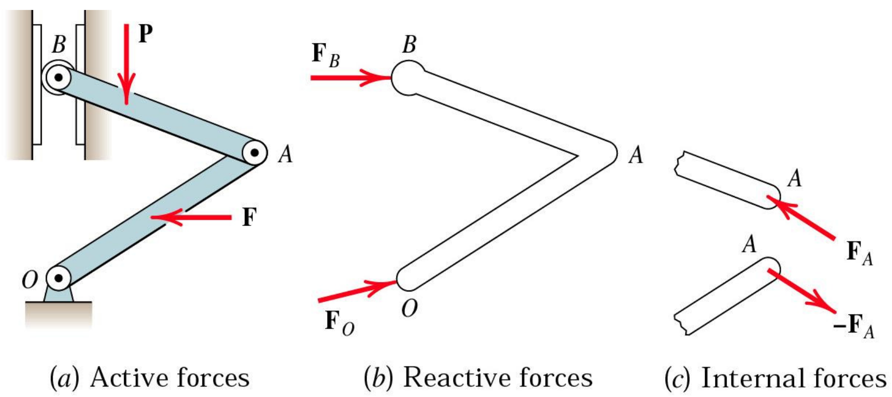

Figure 4

Definition of Virtual Work

Consider the following expression

____

where δU represents the ______________________

$\delta\mathbf{r}$ represents the ______________________

The term ___________________________ is used here to indicate that the displacement ____. It is assumed to exist and the corresponding work done due to the force **F** is given by the

____

The principle of virtual work states that

The virtual work done by external active forces on an ideal mechanical system in equilibrium is zero for any and all virtual displacements consistent with the constraints.

____

Application of the principle of virtual work

We can use the principle of virtual work to __________________________

________________________. This will be illustrated via the following example.

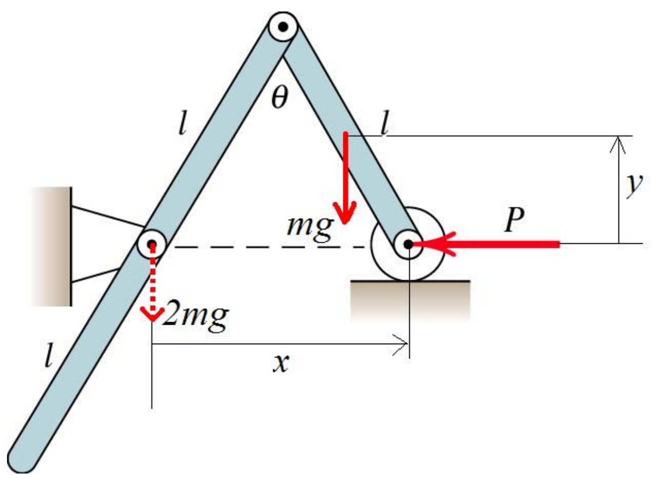

Figure 5

A simple structure consisting of two uniform bars is shown in figure 5. A horizontal force **P** is applied at the roller support. The mass per unit length of the bar is *m/l*.

Use the principle of virtual work to determine _________________________.

1. First, we draw all the active forces on the diagram. This includes **mg** and **P**. Note that **2mg** is acting through the support, hence it can produce no virtual work and is neglected.
2. Define a coordinate system to determine the virtual displacements and virtual work.
   - Let x be the horizontal distance between the fixed support and the roller, so that the virtual work done by force **P** is given by $-P\delta x$. It is negative because the direction of x is opposite to that of force **P**.
   - Let y be the vertical distance between the midpoint of the right hand side bar and the roller support. The virtual work done by force mg is given by $-mg\delta y$.
3. The two quantities of virtual work must be written using a common variable. In this case we will use the angle θ. After some geometry manipulation, the coordinates x and y and their corresponding virtual displacements are transformed as shown below.

Coordinates	Virtual displacement
$x = 2l\sin\frac{\theta}{2}$	$\delta x = l\cos\frac{\theta}{2}\delta\theta$
$y = \frac{l}{2}\cos\frac{\theta}{2}$	$\delta y = -\frac{l}{4}\sin\frac{\theta}{2}\delta\theta$

4. The principle of virtual work states that $\delta U = 0$. We obtain

$$\delta U = 0 = -P\delta x - mg\delta y$$

$$0 = -P\left(l\cos\frac{\theta}{2}\delta\theta\right) - mg\left(-\frac{l}{4}\sin\frac{\theta}{2}\delta\theta\right)$$

$$\theta = 2\tan^{-1}\left(\frac{4P}{mg}\right)$$

Potential Energy and Stability

We will now extend our equilibrium analysis by including the concept

___________________________________.

Elastic potential energy

The simplest example of a device which makes use of ________________ potential energy is a _____________________________. For a spring of linear stiffness k, the force required to compress the spring by a distance x, is given by_______________.

Since the spring has been compressed, work is said to have been done by the force F. The work done on the spring during an infinitesimal displacement dx is___________. Hence the potential energy ________ stored in the spring after the compression is given by

$$V_e = \int_0^x F dx = \int_0^x kx \ dx$$

____

Virtual change in elastic potential energy

The same derivation can be used to determine the _________________________ ______________________which is caused by the virtual displacement (compression) of the spring. Hence a virtual change in elastic potential energy due to a virtual displacement is given by

____

Gravitational potential energy

The gravitational potential energy ________of a body at a vertical distance h above a ______________________ is given by

____

Virtual change in gravitational potential energy

The ____ due to a virtual change in height_________, which is always positive upwards, is given by

____

Energy equation

Statement 1 The work done by a linear spring on the body to which its movable end is attached to is the negative of the change in the elastic potential energy of the spring.

Statement 2 The work done by the gravitational force (or weight) is the negative of the change in gravitational potential energy

Combining the two statements together with the virtual displacement concepts, we can conclude that the sum of the work done by ______________________________ is equal to the work done ____.

The above statement is mathematically represented by

____

where $\delta U'$ represents the ____

(except spring and gravitational)

δV represents the ____,

i.e. ________________________. In another words, it represents the sum of changes in potential energy.

Stability of Equilibrium

Let us consider a system where spring and gravitation forces are present. At equilibrium, there is no resultant force acting on the body. Hence _______________ ____.

____

Using the relationship___________________, it is mathematically equivalent to the requirement

____

This equation only tells us that the total potential energy is ____________________ ___________________ but ____ the system is in. There are three possible scenarios as shown in figure 6.

Figure 6

We can determine the type of stability by using a simple calculus technique similar to determining the maximum, minimum and a neutral point.

Stable equilibrium ____________________

Unstable equilibrium ____________________

Neutral equilibrium ____________________