Lecture 7: Distributed Forces – Part 3: Products of Inertia and Rotation of Axes

Products of Inertia

In certain problems involving (1) _____

and (2) _____,

and expression _____occurs, which can be integrated to give

_____

The quantity I_{xy} is called the _____

_____ .

Note that in previous lecture moments of inertia about the x- and y-axes are called I_x and I_y, respectively. Sometimes these quantities are written I_{xx} and I_{yy}, in order to be consistent with the I_{xy} notation.

Transfer of Axes

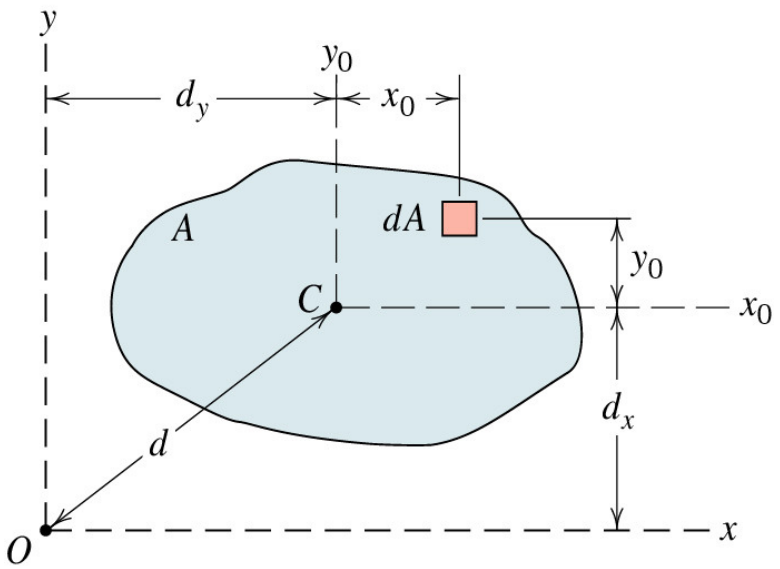

Figure 1

The axes of the product of inertia can also be transferred in a similar fashion to the _____. Since the product of inertia depends on both x- and y-

axes, the transfer of axis theorem for products of inertia for the shape shown in figure 1 is given by

_____

where $\bar{I}_{xy}$ is the product of inertia about the centroidal axes, d_x and d_y are the distances that the x-and y-axes have shifted, respectively.

Rotation of Axes

The product of inertia is required when we need to calculate the moment of inertia _____.

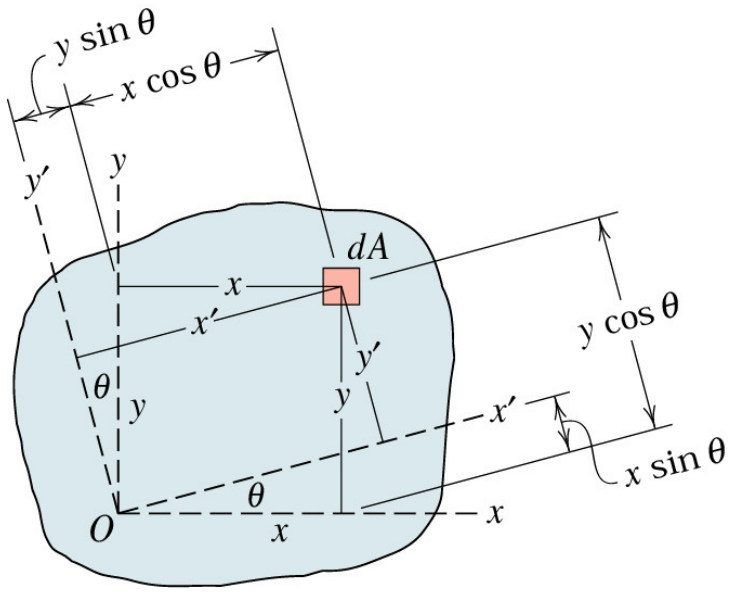

Figure 2

Figure 2 shows two sets of axes whose origins coincide on an irregularly shaped lamina. The new x' and y' axes _____ from the original x-y axes.

The moments of inertia about the rotated $x' - y'$ axes are given by

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \qquad \textbf{(1)}$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \qquad \textbf{(2)}$$

Similarly the product of inertia about the inclined axes is given by

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \qquad \textbf{(3)}$$

Maximum and Minimum Moments of Inertia

The magnitudes of the moments of inertia about the rotated axes vary with the rotated angle θ. Hence the must exist a critical angle where the maximum or minimum value of the moment of inertia is found.

We will use a simple method from basic calculus to determine the maximum or minimum moment of inertia. By differentiating $I_{x'}$ with respect to the angle θ, and equating it to zero, we obtain

$$\frac{dI_{x'}}{d\theta} = \left(I_y - I_x\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

This expression simplifies to

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \qquad \textbf{(4)}$$

where α represents the critical angle.

Mohr's Circle of Inertia

This section will introduce another method called the _____ to determine the _____ values of moments of inertia.

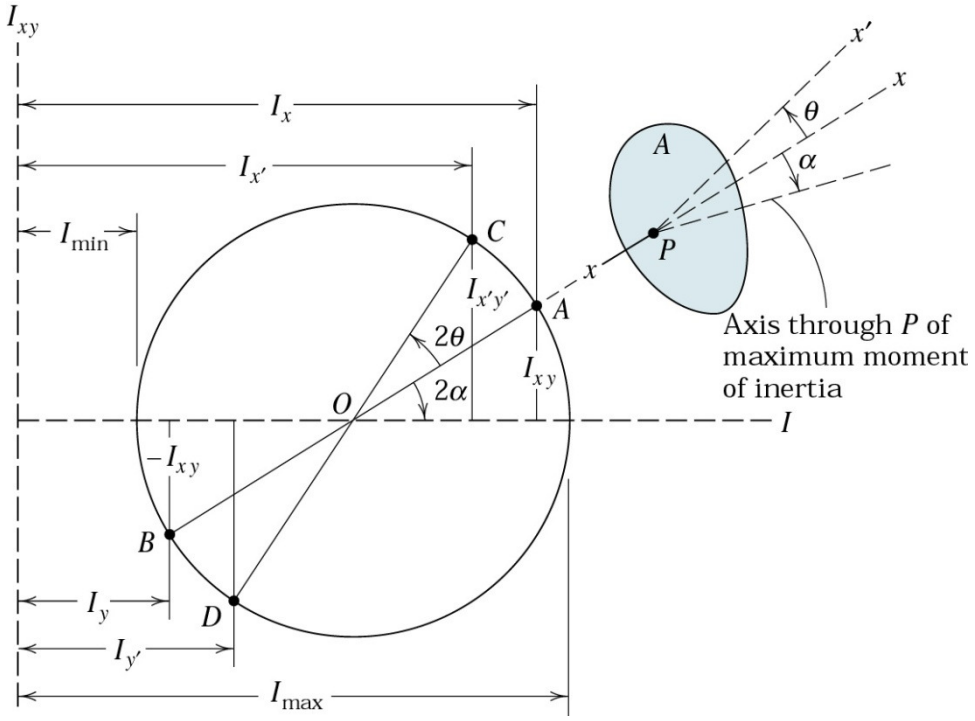

Figure 3

An example of the Mohr's circle is shown in figure 3. The circle shows moments of inertia of the shaded area *A* shown on the right hand side.

How do we construct the Mohr's circle?

1. The values of I_x, I_y and I_{xy} must first be defined.
2. Draw the axes using *I* as the horizontal axis and I_{xy} as the vertical axis.
3. Plot point *A* whose coordinates are (I_x, I_{xy})
4. Plot point *B* whose coordinates are $(I_y, -I_{xy})$
5. Draw a straight line *AB*. Point *O* is located at the intersection between *AB* and the horizontal axis.
6. Draw a circle whose origin is at point *O* and the length of its radius is the distance *OA* (or *OB*), i.e. points *A* and *B* are on the circumference of the circle.

How do we use the Mohr's circle to determine $I_{x'}, I_{y'}$ and $I_{x'y'}$***?***

Points A and B indicate the values of original I_x, I_y and I_{xy}. Suppose we want to rotate the *x*-*y* axes in the _____, see right hand side of figure 3. The new moments of inertia are now $I_{x'}, I_{y'}$ and $I_{x'y'}$, and the values of these quantities can be directly determined from the Mohr's circle by drawing a diameter to the circle whose inclination is at an _____ _____ from *AB*.

How do we use the Mohr's circle to determine $I_{\max}$ **and** $I_{\min}$ ***?***

From figure 3, the points where the circle circumference intersects the horizontal axis are the locations of the maximum and minimum moments of inertia. In this case, the line OA is at an _____, which means that the axis on the shaded area which corresponds to the maximum moment of inertia is at _____ _____.

Remarks on the Maximum and Minimum Moment of Inertia

The axes where the maximum and minimum moments of inertia are found are called the _____.